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ON THE LOGIC OF QUANTUM LOGIC

We want to clarify the connection between the usual logic and the so-called "Quantum Logic", or to be more precise, we want to interpret the structure of the propositional system and compare this structure with the formal logic. Let us first recall the definition of what we call questions, propositions and proporties.

We take the realistic point of view. The system is and it is what it is. It has different properties and whether these properties are known or not by somebody does not change anything to the reality itself, We have to describe these properties and not to explain how the physicist can increase his information about the system. In fact when the physicist believes that the system possess a certain property he checks up by performing an experiment. If he obtains what he expects he is reinforced in his belief but if he obtains something else he must admit his error and change his belief. We shall define a property by the corresponding test. Thus, in order to be precise let us introduce the following concepts.

We shall call *question* any experiment leading to an alternative with terms "yes" and "no". An ideal yes–no measurement is a question, however there are many questions which are not ideal and not measurements. When we may affirm with certainty that in the case of an experiment the result would be "yes" we shall say that the *question is true* for the system as it actually is. It is a property of the system which of course may disappear during the evolution.

The set of all questions true for an actual given system defines its *state* and in conclusion of the remark just before, the state will change during the evolution.

We shall say that the question α is *stronger* than the question β if β is true each time α is true. This is a relation which expresses a physical law and we shall write:

$$\alpha < \beta.$$

This preordering relation defines an equivalence relation and we shall call

Journal of Philosophical Logic 6 (1977) 481–484. *All Rights Reserved. Copyright © 1977 by D. Reidel Publishing Company, Dordrecht-Holland.*

proposition an equivalence class of question. A *proposition* is *true* if one of the questions in the equivalence class is true and so by difinition the others are also true. We shall identify a true proposition with an *actual property* of the system, any other proposition being only a *potential property*.

The set of all possible propositions (true or not) defined for a system and equipped with the partial order relation "$<$" is a complete lattice $\mathfrak{L}$. To prove this theorem we must exhibit for any subset of propositions a_i the corresponding proposition $\bigwedge a_i$, the greatest lower bound, and also $\bigvee a_i$, the least upper bound. We first prove the existence of $\bigwedge a_i$. For each i let us choose a question α_i in the equivalence class a_i. The α_i being given we may define a new question called $\pi\alpha_i$ according to the following prescription:

To perform the question $\pi\alpha_i$, start by choosing freely (or at random) one i, then perform the corresponding experiment, and finally attribute to $\pi\alpha_i$ the result ("yes" or "no") just obtained.

In this condition it is clear that $\pi\alpha_i$ is true if and only if α_i is true for every i. In other words we have the relation:

$$\alpha < \pi\alpha_i \iff \alpha < \alpha_i \forall\, i$$

and the equivalence class defined by $\pi\alpha_i$ is really the greatest lower bound of the a_i.

Since there exists a trivial proposition I (the maximal element) we can define $\bigvee a_i$ by:

$$\bigvee a_i = \bigwedge x \quad a_i < x \forall i$$

Thus, from this proof follows immediately that the relation corresponds to the "and":

"$a \wedge b$ is true" iff "a is true" and "b is true";

however the relation $\vee$ does not correspond to the "or"; we have only:

"$a \vee b$ is true" if "a is true" or "b is true".

The state S of the system is completely defined by the true proposition $p = \bigwedge x$, $x \in S$, since according to our definition:

$$S = (x \mid p < x \text{ and } x \in \mathfrak{L})$$

Here, to translate the idea of completeness of this lattice description, we make the hypothesis that p is an atom. From this postulate follows the

atomicity of $\mathfrak{L}$: if a is a proposition different from the trivial one 0 (the minimal element), a is true for same state and, there exists an atom $p < a$.

THEOREM: The relation $\vee$ corresponds to the "or" if and only if $\mathfrak{L}$ is distributive.

Proof: The necessity comes directly from the rules of the formal logics. On the other hand if $a \vee b$ is true for the state p we have $p < a \vee b$ and then, if $\mathfrak{L}$ is distributive, $p < a$ or $p < b$, which means "a is true" or "b is true".

Now we will define a', a *compatible complement* for a, as a complement ($a' \vee a = I, a' \wedge a = 0$) such that there exists a question $\alpha \in a$ with $\tilde{\alpha} \in a'$ where by definition $\tilde{\alpha}$ is the dual of α obtained by exchanging in the alternative the "yes" and the "no". Then we make a new hypothesis by supposing the existence of at least one compatible complement for each proposition.

THEOREM: If for each state and each proposition a we have "a is true" or "a' is true" then $\mathfrak{L}$ is Boolean and isomorphic to the lattice of all the subset of a set.

Proof: Following the hypothesis it is clear that a' is the orthocomplement of a.[1] Moreover $\mathfrak{L}$ is weakly modular since "b is true" and $a < b$ implies "$a \vee (b \wedge a')$ is true", i.e., $a < b$ implies $b < a \vee (b \wedge a')$. And finally for each pair of proposition a and b we have:

$$(a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b') = I$$

hence a is compatible with b. From this follows that $\mathfrak{L}$ is Boolean. Since $\mathfrak{L}$ is complete and by hypothesis atomic, it is isomorphic to a Boolean lattice of all subset of a set.

By this theorem we have put in evidence the *a priori* which is the fundamental principle of the classical description. Since there are only two possible results "yes" and "no" the classical physicist has always inferred the necessity of "a is true" or "a' is true". This is a logical mistake: we cannot call up such principles as causality or completeness to claim that one of the two possible results is *a priori* certain, since the experiment itself is only hypothetical. For this reason in general quantum physics we replace the Boolean axiom by two weaker assumptions:

(1) The orthomodularity:

$$a < b \Rightarrow a' = b' \vee (a' \wedge b)$$

a' and b' compatible complements for a and b,

(2) The covering law:

p is an atom and $a \wedge p = 0 \Rightarrow (a \vee p) \wedge a'$ is an atom.

Let us conclude by a remark. If we know that the state is such that "a is true" or "b is true" (this is not always the case even if "$a \vee b$ is true"!) we cannot in general remove this alternative in one experiment unless $b < a'$. Indeed if $b < a'$ we can very well remove the alternative by performing a question α such that $\alpha \in a$ and $\tilde{\alpha} \in a'$.

NOTE

[1] For all definitions in lattice theory see: C. Piron, *Foundations of Quantum Physics*, W. A. Benjamin, Inc., Advanced Book Program, Massachusetts, U.S.A., 1976.